On optimal allocation in coherent systems: A review with an emphasis on some recent developments

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Abstract It is of great interest to allocate redundant component(s) in a coherent system in order to optimize the lifetime of the resulting system in reliability engineering, and system security. This topic has posed many interesting theoretical problems to which many researchers have devoted themselves in the past decades. In this article, we aim to review some recent results on the problem of optimal allocations of active [standby] redundancies in coherent systems. Some open problems in this research line are posed as well. Here we highlight the stochastic orderings as a powerful tool in our discussion.

Keywords Likelihood ratio order; Hazard rate order; Allocation; Coherent system
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1 Introduction

In system reliability engineering, systems are made up of different components, e. g. , infrastructure networks, supercomputers and nuclear reactors. One particular type of system is the coherent system which is a system composed of a number of components that fails as soon as the set of failed components reaches certain fatal set-thresholds. Consider a coherent binary system (Barlow and Proschan [1]) consisting of n-components in which each component and the system can be in two performance levels: working or failed. A system made up of n-components is coherent if it satisfies the following conditions: (i) every component is relevant, i. e. , each component contributes to the functioning/failure of the system; and (ii) the reliability of the system is monotonic, i. e. , if a failed component is removed, then the reliability of the system can only increase.

System safety is one of the main concerns in system reliability engineering, especially for systems that required high reliability such as nuclear reactors. It has been shown that redundancy allocation technique can be used to improve the reliability of the system. In this regard, it is of great interest to allocate redundant component(s) in a system with the aim of optimizing the lifetime of the resulting system in reliability engineering and system safety. This topic was first investigated by Boland, El-Newwihi and Prochan [4]. They studied both active redundancy and standby redundancy and proved a general result in allocating a redundancy in a k-out-of-n system. Shaked and Shanthikumar [5] also studied a similar problem using the majorization approach. They considered the problem of allocating K redundancies to a series system. Since then, a lot of researchers have been working on this problem, e. g. , Singh and Misra [30], Singh and Singh [31,32], Valdes and Zequeira [34,35], Valdes et al. [33], Brito et al. [8], Hu and Wang [14], Misra et al. [22—24], Li and Ding [15], Zhao et al. [36], Ding and Li [11], Zhao et al. [37,38], Zhao

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et al. [39] and the references therein.

In general, there are two types of redundancies called active redundancy and standby redundancy commonly used in reliability engineering and system security. The former [active redundancy] is used when replacement of components during the operation of the system is impossible; in this case the redundancies are put in parallel to components of the system, which leads to taking the maximum of random lifetimes. The latter [standby redundancy] is used when replacement of components during the operation of the system is possible; in this case the redundancy starts functioning immediately after the corresponding original component in the system fails, which leads to taking the convolution of random variables.

For the sake of convenience, we first recall some pertinent notations of stochastic orders, and majorization and related orders. Throughout this paper, the term increasing is used for monotone non-decreasing and decreasing is used for monotone non-increasing.

1.1 Stochastic orders

Definition 1.1 For two random variables X and Y with densities f_X and f_Y, and distribution functions F_X and F_Y, respectively, let $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$ be the corresponding survival functions. Then: X is said to be smaller than Y in the

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f_X(x)/f_Y(x)$ is increasing in x;

(ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{F}_X(x)/\bar{F}_Y(x)$ is increasing in x;

(iii) reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $F_X(x)/F_Y(x)$ is increasing in x;

(iv) stochasticorder (denoted by $X \leq_{st} Y$) if $\bar{F}_X(x) \geq \bar{F}_Y(x)$ for all x;

(v) stochastic precedence order (denoted by $X \leq_{sp} Y$) if $P(X > Y) \leq P(X > Y)$;

(vi) increasing convex order (denoted by $X \leq_{icx} Y$) if $\int_t^\infty \bar{F}_X(x)\mathrm{d}x \leq \int_t^\infty \bar{F}_Y(x)\mathrm{d}x$ for all $t \geq 0$;

(vii) increasing concave order (denoted by $X \leq_{icv} Y$) if $\int_0^t \bar{F}_X(x)\mathrm{d}x \leq \int_0^t \bar{F}_Y(x)\mathrm{d}x$ for all $t \geq 0$;

(viii) dispersive order (denoted by $X \leq_{disp} Y$) if $F_1^{-1}(v) - F_1^{-1}(u) \leq F_2^{-1}(v) - F_2^{-1}(u)$ for $0 \leq u \leq v \leq 1$, where F_1^{-1} and F_2^{-1} are the right continuous inverses of the distribution function of X and Y.

It is known that the likelihood ratio order implies both the hazard rate order and the reversed hazard rate order which in turn implies the usual stochastic order, but neither the hazard rate order nor the reversed hazard rate order implies the other. One may refer to Blyth [2], Boland et al. [6], Shaked and Shanthikumar [29], and Muller and Stoyan [25] for details of various stochastic orders.

1.2 Majorization and related orders

The notion of majorization is quite useful in establishing various inequalities. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of the components of the vector$x = (x_1, \cdots, x_n)$.

Definition 1.2 The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$, written as $\boldsymbol{x} \geq_m \boldsymbol{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \text{ for } j = 1, \cdots, n-1,$$

and $\sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}$.

Themajorization relation $\boldsymbol{x} \geq_m \boldsymbol{y}$ means the components of $\boldsymbol{x}$ are more equal than those of $\boldsymbol{y}$ (cf. Marshall and Olkin [19]). In addition, the vector $\boldsymbol{x}$ is said to submajorize the vector $\boldsymbol{y}$ weakly, written as $\boldsymbol{x} \geq_w \boldsymbol{y}$, if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \text{ for } j = 1, \cdots, n.$$

Clearly,

$$\boldsymbol{x} \geq_m \boldsymbol{y} \Rightarrow \boldsymbol{x} \geq_w \boldsymbol{y}.$$

For an extensive and comprehensive discussion on the theory and applications of the majorization order, one may refer to Marshall and Olkin [19]. Bon and Paltanea [7] introduced a pre-order on $\mathfrak{R}_+^n$, called p-larger order, which is defined as follows.

Definition 1.3 The vector $\boldsymbol{x}$ in $\Re_+^n$ is said to be p-larger than another vector $\boldsymbol{y}$, written as $\boldsymbol{x} \geq_p \boldsymbol{y}$, if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \text{ for } j = 1, \cdots, n.$$

Let $\log\boldsymbol{x}$ be the vector of logarithms of the coordinates of x. It is then easy to verify that

$$\boldsymbol{x} \geq_p \boldsymbol{y} \Leftrightarrow \log\boldsymbol{x} \geq_w \log\boldsymbol{y}.$$

Moreover,

$$\boldsymbol{x} \geq_w \boldsymbol{y} \Rightarrow \boldsymbol{x} \geq_p \boldsymbol{y}$$

for $\boldsymbol{x}, \boldsymbol{y} \in \Re_+^n$. The converse is, however, not true. For example, we have $(2,7) \geq_p (3,5)$, but the weak majorization order clearly does not hold between these two vectors.

2 Active redundancy

Consider a coherent system consisting of components $C_1, \cdots, C_n$ with respective random lifetimes $X_1, \cdots, X_n$. We now have redundancies $R, R_1, \cdots, R_k$ with respective random lifetimes $X, Y_1, \cdots, Y_k$. Assume all the nonnegative random lifetimes $X_1, \cdots, X_n, X, Y_1, \cdots, Y_k$ are statistically independent.

2.1 Allocation of one or two active redundancies

We first introduce the active redundancy models. One can decide which of these two allocations is better by conducting stochastic comparisons on

$$T_1 = \min\{\max\{X_1, X\}, X_2, \cdots, X_n\}$$

and

$$T_2 = \min\{X_1, \max\{X_2, X\}, X_3, \cdots, X_n\}$$

for the series system. Note that in this case it is unnecessary to consider the parallel system. Some results in the references focus on the allocation of one active redundancy in a two-component series system by stochastically comparing

$$S_1 = \min\{\max\{X_1, X\}, X_2\} \text{ and } S_2 = \min\{X_1, \max\{X_2, X\}\}.$$

Also, two models which are mathematically more general than above one. In the first one, we have two spares R_1 and R_2 (possibly identical) with respective random lifetimes Y_1 and Y_2. Due to some certain constraints, we can only use one of them, either R_1 in C_1, or R_2 in C_2. To decide which of these two allocations is better, one can make stochastic comparisons on

$$M_1 = \min\{\max\{X_1, Y_1\}, X_2, \cdots, X_n\}$$

and

$$M_2 = \min\{X_1, \max\{X_2, Y_2\}, X_3, \cdots, X_n\}$$

for the series system. For the two-component series system, we define

$$N_1 = \min\{\max\{X_1, Y_1\}, X_2\}$$

and

$$N_2 = \min\{X_1, \max\{X_2, Y_2\}\}.$$

Note that if Y_1 and Y_2 have identical distribution, then this model reduces to the case of allocation of one active redundancy mentioned above.

In the second model, we have two spares R_1 and R_2 which can be used in one of the following two allocation ways: R_1 with C_1 and R_2 with C_2, or R_1 with C_2 and R_2 with C_1. One can compare these two allocation ways through stochastic comparisons on

$$H_1 = \min\{\max\{X_1, Y_1\}, \max\{X_2, Y_2\}, X_3, \cdots, X_n\}$$

and

$$H_2 = \min\{\max\{X_1, Y_2\}, \max\{X_2, Y_1\}, X_3, \cdots, X_n\}$$

for the series system, and for the two-component case, define

$$J_1 = \min\{\max\{X_1, Y_1\}, max\{X_2, Y_2\}\}$$

and

$$J_2 = \min\{\max\{X_1, Y_2\}, \max\{X_2, Y_1\}\}.$$

Note if $P(Y_2 = 0) = 1$, then this model reduces to the model of allocation of one active redundancy.

We now review some results in the literature along with this research line and the related references include Boland, El-Neweihi and Proschan [4,5], Singh and Misra [30], Mi [21], Valdes and Zequeira [34,35], Romera et al. [27], Li and Hu [16], Li, Yan and Hu [17], Valdes et al. [33], Brito et al. [8] and Zhao et al. [36—38].

Boland et al. [5] proved that

$$X_1 \leq_{st} X_2 \Leftrightarrow \{\max\{X_1, X\}, X_2, \cdots, X_n\}_{[k]} \geq_{st} \{X_1, \max\{X_2, X\}, X_3, \cdots, X_n\}_{[k]},$$

which is a general result for k-out-of-n system and implies that $X_1 \leq_{st} X_2 \Leftrightarrow T_1 \geq_{st} T_2$. In many situations, one may want to consider the joint distributions, rather than restricting attention to their marginal distributions. The precedence order takes the joint distribution into consideration. Singh and Misra [30] showed that

$$X_1 \leq_{st} X_2 \Rightarrow S_1 \geq_{sp} S_2,$$

and they also showed with the help of a counterexample that the converse of the above result does not hold. In fact, they obtained a more general result for the k-out-of-n system as below.

$$X_1 \leq_{st} X_2 \Rightarrow \{\max\{X_1, X\}, X_2, \cdots, X_n\}_{[k]} \geq_{sp} \{X_1, \max\{X_2, X\}, X_3, \cdots, X_n\}_{[k]}.$$

Li and Hu[16] proved that

$$X_1 \leq_{icv} X_2 \Rightarrow S_1 \leq_{icv} S_2,$$

and if X and X_1 (or X_2) have convex survival functions, then

$$X_1 \leq_{icv} X_2 \Rightarrow S_1 \leq_{sp} S_2.$$

Singh and Misra [30] also made comparison between S_1 and S_2 in terms of the hazard rate ordering when X_1, X_2 and Y have independent exponential distributions, i. e. , if X_1, X_2 and Yhave independent exponential distributions with parameters λ_1, λ_2 and λ, respectively, then

$$\lambda_1 \geq \max\{\lambda_2, \lambda\} \Rightarrow S_1 \geq_{hr} S_2. \quad (1)$$

Zhao et al. [36] strengthened the result in (1) from the hazard rate order to the likelihood ratio order in the following theorem.

Theorem 2.1 Let X_1, X_2, and X be independent exponential random variables with parameters λ_1, λ_2 and λ, respectively. Denote $S_1 = \min\{\max\{X_1, X\}, X_2\}$ and $S_2 = \min\{X_1, \max\{X_2, X\}\}$. If $\lambda_1 \geq max \{\lambda_2, \lambda\}$, then $S_1 \geq_{lr} S_2$.

Valdes and Zequeira [34] proved that if $X_1 \leq_{hr} X_2$, $X_1 \leq_{hr} Y_1$, $Y_2 \leq_{hr} Y_1$ and the hazard rate ratio $r_2(t)/r_1(t)$ of X_2 and X_1 is decreasing in t , then $N_1 \geq_{hr} N_2$. Misra et al. [23] further improved the above result and obtained if $X_1 \leq_{st} X_2$, $X_1 \leq_{hr} Y_1$, $Y_2 \leq_{hr} Y_1$ and $r_1(t)\bar{F}_2(t) \geq r_2(t)\bar{F}_1(t)$, then $N_1 \geq_{hr} N_2$. Li et al. [17] recently investigated the reversed hazard rate order in this setup and proved that if $X_1 \leq_{rh} X_2$, $Y_2 \leq_{rh} Y_1$ and $\bar{G}_1(t) G_2(t) F_2(t) \geq \bar{G}_2(t) G_1(t) F_1(t)$, then $N_1 \geq_{hr} N_2$. Misra et al. [23] got another sufficient condition for the reversed hazard rate order: ' $X_1 \leq_{rh} X_2$, $X_1 \leq_{rh} Y_1$, $Y_2 \leq_{rh} Y_1$, and $F_1(t) G_1(t) \geq F_2(t) G_2(t)$ '. It seems that this sufficient condition is more feasible. As pointed out by Misra et al. [23], the condition $F_1(t) G_1(t) \geq F_2(t) G_2(t)$ is actually equivalent to $\max(X_1, Y_1) \leq_{st} \max(X_2, Y_2)$. On the other hand, Valdes and Zequeira [34] also provided two sufficient conditions for the usual stochastic order to hold between N_1 and N_2 , that is, (a) $X_1 \leq_{st} X_2$, $Y_2 \leq_{st} Y_1$; (b) $X_1 \leq_{st} X_2$, $X_1 \leq_{st} Y_1$, $Y_2 \leq_{hr} X_2$. For the likelihood ratio order, Zhao et al. [38] obtained the following new result.

Theorem 2.2 Let X_1, X_2, Y_1 *and* Y_2 be independent exponential random variables with parameters λ_1, λ_2, λ_1^* and λ_2^* , respectively. If either $\lambda_2 \leq \lambda_1^* \leq \lambda_1 \leq \lambda_2^*$ and $(\lambda_2, \lambda_2^*) \geq_w (\lambda_1, \lambda_1^*)$, or $\lambda_2 \leq \lambda_1^* \leq \lambda_2^* \leq \lambda_1$ and $(\lambda_2, \lambda_1) \geq_w (\lambda_2^*, \lambda_1^*)$, then $N_1 \leq_{lr} N_2$

It is also of interest for the case when $X_i =_{st} Y_i, i = 1, 2$. Proposition 3 in Valdes and Zequeira [34] claimed that there is no hazard rate order between N_1 and N_2 under the condition $X_1 \leq_{hr} X_2$, but we have the following result for reversed hazard rate order under the exponential setup.

Theorem 2.3 Let X_1, X_2, Y_1 and Y_2 be independent exponential random variables with parameters λ_1, λ_2, λ_1 and λ_2 , respectively. If $\lambda_1 \geq \lambda_2$, then $N_1 \geq_{rh} N_2$.

Romera et al. [27], Li et al. [17] and Misra et al. [23] investigated the stochastic precedence order between M_1 and M_2 . Specifically, Romera et al. (2004) proved that if $X_1 \leq_{st} X_2$ and $\bar{F}_2(t)\bar{G}_1(t) \geq$

$\bar{F}_1(t)\bar{G}_2(t)$, then $M_1 \geq_{sp} M_2$. Li et al. [17] and Misra et al. [23] independently proved that if $X_1 \leq_{icv} X_2$, $Y_2 \leq_{icv} Y_1$ and $Y_2, X_1, X_3, X_4, \cdots, X_n$ (or $Y_1, X_2, X_3, X_4, \cdots, X_n$) has convex survival functions, then $M_1 \geq_{sp} M_2$.

Valdes et al. [33] proved that $M_1 \geq_{icv} M_2$ holds provided either of the following two conditions holds: (a) $X_1 \leq_{icv} X_2$, $Y_2 \leq_{icv} Y_1$; or (b) $X_1 \leq_{icv} X_2$, $X_1 \leq_{st} Y_1$ and $Y_2 \leq_{st} X_2$. Misra et al. [23] generalized/supplemented the results of Valdes et al. (2010) as below. If any one of the following conditions holds: (a) $X_1 \leq_{icv} X_2$ and $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t)$; (b) $Y_1 \leq_{icv} Y_2$ and $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t)$; (c) $X_1 \leq_{icv} X_2$ and $F_2(t)\, G_2(t) \geq F_1(t)\, G_1(t)$, then $M_1 \geq_{icv} M_2$.

Boland et al. [4] proved that if $X_1 \leq_{st} X_2$ and $Y_1 \leq_{st} Y_2$, then $H_1 \leq_{st} H_2$. Romera et al. [27] proved if $X_1 \leq_{hr} X_2$ and $Y_2 \leq_{hr} Y_1$, then $H_1 \geq_{sp} H_2$. Boland et al. [4] and Romera et al. [27] also proved similar results for the k-out-of-n systems. Valdes et al. [33] showed $H_1 \geq_{sp} H_2$ also holds under the condition $X_1 \leq_{rh} X_2$ and $Y_2 \leq_{rh} Y_1$. Moreover, they proved if $X_i =_{st} Y_i, i=1,2$, and $X_1 \leq_{rh} [\leq_{hr}]\, X_2$ or $X_2 \leq_{rh} [\leq_{hr}]\, X_1$, then $H_1 \leq_{sp} H_2$. Misra et al. [23] listed some sufficient conditions for the stochastic precedence order.

Valdes and Zequeira [35] proved that if $X_i =_{st} Y_i, i=1,2$, $X_1 \leq_{hr} X_2$, and the hazard rate ratio $r_2(t)/r_1(t)$ of X_2 and X_1 is decreasing, then $J_1 \leq_{hr} J_2$; while Brito et al. [8] proved that if $X_i =_{st} Y_i, i=1,2$, $X_1 \leq_{rh} X_2$, and the hazard rate ratio $r_1(t)/r_2(t)$ of X_1 and X_2 is increasing, then $J_1 \leq_{rh} J_2$. Misra et al. [23] improved the result of Valdes and Zequeira [35] and proved if $X_i =_{st} Y_i, i=1,2$, $X_1 \leq_{st} X_2$, and $r_1(t)F_2(t) \geq r_2(t)F_1(t)$, then $J_1 \leq_{hr} J_2$. In this regard, Zhao et al. [37] established the following result for the likelihood ratio order.

Theorem 2.4 Let X_1, X_2, Y_1 and Y_2 be independent exponential random variables with parameters λ_1, λ_2, λ_1 and λ_2 , respectively. Then, $J_1 \leq_{lr} J_2$.

At the end of this subsection, we list two open problems that have not been solved yet. The first one is involved in the dispersive order.

Problem 2.1 Let X_1, X_2 and X be independent exponential random variables with parameters λ_1, λ_2 and λ , respectively. Denote $S_1 = \min\{\max\{X_1, X\}, X_2\}$ and $S_2 = \min\{X_1, \max\{X_2, X\}\}$. If $\lambda_1 \geq \max\{\lambda_2, \lambda\}$, then $S_1 \geq_{disp} S_2$.

Problem 2.2 Let X_1, X_2, Y_1 and Y_2 be independent exponential random variables with parameters λ_1, λ_2, λ_1^* and λ_2^* , respectively. Under what condition it holds $J_1 \geq_{lr} J_2$.

2.2 Allocation of *k* active redundancies

Suppose we have a general coherent system φ consisting of n components $C_1, \cdots, C_n$ having independent lifetimes $X_1, \cdots, X_n$ with respective distribution functions $F_1, \cdots, F_n$. Denote $T(\boldsymbol{X}) = \tau(X_1, \cdots, X_n)$ as the lifetime of the coherent system $\emptyset$. k active spares $R_1, \cdots, R_k$ have independent and identically distributed lifetimes $Y_1, \cdots, Y_k$ with common distribution function G , and $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_k)$ are statistically independent. We are now going to discuss the problem of allocating these k active spares to$n$ original components. Any element of the set

$$\mathbb{K} = \left\{L = (l_1, \cdots, l_n): l_i = 0,1,\cdots,k, i = 1,\cdots,n, \sum_{i=1}^{n} l_i = k\right\}$$

is said to be an allocation. Denote $T_S(K), K \in \mathbb{K}$, the lifetime of the resulting system obtained from the series system $\emptyset_S$ by putting K_i spares to C_i , $i=1,\cdots,n$. It should be mentioned here that all allocations in a parallel system will yield the same life distribution and thus we do not need to consider this case.

Suppose all X_i and Y_i are i. i. d. , in other words, $F_i = F = G$. Shaked and Shanthikumar [28] considered the problem of allocating k redundancies to a series system and established that

$$T_S(K) \geq_{st} T_S(K') \quad \text{whenever} \quad K' \geq_m K, \tag{2}$$

which can also be obtained from Theorem 3. 3 of Hu and Wang [14] or Theorem 2. 2 of Boland et al. [4]. In fact, Hu and Wang [14] got this result for any k-out-of-n system, while Boland et al. [4] obtained a more general result wherein even the condition that all rvs are i. i. d. is unnecessary. Singh and Singh [31] further improved the result in (2) from the usual stochastic order to the hazard rate order as

$$T_S(K) \geq_{hr} T_S(K') \text{ whenever } K' \geq_m K. \quad (3)$$

Misra et al. [22] showed by a counterexample (Example 2.2) that the result in (3) may not hold without the condition that all rvs are i. i. d. Ding and Li [11] recently established that

$$T_{k|n}(K) \geq_{hr} T_{k|n}(K') \text{ whenever } K' \geq_m K, \quad (4)$$

which extends the result in (3) by Singh and Singh [31] from the series case to the k-out-of-n case and also strengthens the result by Hu and Wang [14] from the usual stochastic order to the hazard rate order. As an application of the result of (4), Ding and Li [11] showed that, for two non-negative random vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ with their hazard rates proportional to a common baseline one by rational parameters vector $(\lambda_1, \cdots, \lambda_n)$ and $(\mu_1, \cdots, \mu_n)$,

$$X_{k:n} \geq_{rh} Y_{k:n} \quad \text{whenever} \quad (\lambda_1, \cdots, \lambda_n) \geq_m (\mu_1, \cdots, \mu_n),$$

thereby strengthening the result by Pledger and Proschan [26] from the usual stochastic order to reversed hazard rate order. Here, we pose the following two open problems for order statistics from the PHR sample or the gamma sample.

Problem 2.3 For two non-negative gamma random vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ with common parameter r and scale parameters vector $(\lambda_1, \cdots, \lambda_n)$ and $(\mu_1, \cdots, \mu_n)$,

$$X_{k:n} \geq_{rh} Y_{k:n} \quad \text{whenever} \quad (\lambda_1, \cdots, \lambda_n) \geq_m (\mu_1, \cdots, \mu_n).$$

Problem 2.4 For two non-negative gamma random vectors $(X_1, \cdots, X_n)$ and $(Y_1, \cdots, Y_n)$ with common parameter r and scale parameters vector $(\lambda_1, \cdots, \lambda_n)$ and $(\mu_1, \cdots, \mu_n)$,

$$X_{k:n} \geq_{rh} Y_{k:n} \quad \text{whenever} \quad (\lambda_1, \cdots, \lambda_n) \geq_m (\mu_1, \cdots, \mu_n).$$

For the special case when $n = 2$, Hu and Wang [14] and Misra et al. [22] independently proved that, for a series system with two nodes,

$$T_S(k_1, k_2) \geq_{rh} T_S(k_1', k_2') \text{ whenever } (k_1', k_2') \geq_m (k_1, k_2). \quad (5)$$

Hu and Wang [14] also used a counterexample to show that (5) does not hold in general for the case when $n > 2$. However, they left an open problem whether the result in (5) may be strengthened to likelihood ratio order. Zhao et al. [36] solved this problem and obtained that

$$T_S(k_1, k_2) \geq_{lr} T_S(k_1', k_2') \text{ whenever } (k_1', k_2') \geq_m (k_1, k_2).$$

In fact, they reached a more general result as

$$T_S(k_1, k_2) \geq_{lr} T_S(k_1', k_2') \text{ whenever } (k_1', k_2') \geq_w (k_1, k_2).$$

They also established that

$$T_S(k_1, k_2) \geq_{rh} T_S(k_1', k_2') \text{ whenever } (k_1'+1, k_2'+1) \geq_p (k_1+1, k_2+1),$$

which allows the reliability engineer to obtain more reliable resulting system even though he/she has less redundancies. For the n-components series system, it is shown if $K = nk = \sum_{i=0}^{n} k_i$ and $K_0 = (k, \cdots, k)$, then

$$T_S(K_0) \geq_{lr} T_S(K).$$

And if $(k^* + 1)^n = \prod_{i=0}^{n} (k_i + 1)$ and $K^* = (k^*, \cdots, k^*)$, then

$$T_S(K^*) \geq_{rh} T_S(K). \quad (6)$$

They also showed by a counter example that the reversed hazard rate order in (6) cannot be replaced by the hazard rate order.

Suppose $F_i = F \neq G$. Misra et al. [22] proved, if $\log(G(x))/\log(F(x))$ is an increasing function on $\Re_+$, then

$$T_s(K) \geq_{hr} T_s(K') \quad \text{whenever} \quad K' \geq_m K.$$

Since $r_G(x)/\ r_F(x)$ (r is failure rate funtion) is an increasing function of x on $\Re_+$ implies that $\log(G(x))/\log(F(x))$ is an increasing function, it is also a sufficient condition and seems to be easily verified. Li and Ding [15] further considered this problem for the k-out-of-n system. Denote $T_{k|n}(K)$, $K \in \mathbb{K}$ as the lifetime of the resulting system obtained from a k-out-of-n system by putting K_i spares to C_i,

$i = 1, \cdots, n$. They proved that

$$T_{k|n}(K) \geq_{st} T_{k|n}(K') \quad \text{whenever} \quad K' \geq_m K.$$

Based on the result on the series system, we have the following open problem for the general k-out-of-n system.

Problem 2.5 If log $(G(x))/\log(F(x))$ is an increasing function on $\Re_+$, then

$$T_{k|n}(K) \geq_{hr} T_{k|n}(K') \quad \text{whenever} \quad K' \geq_m K.$$

Suppose $F_n \leq_{st} F_{n-1} \leq_{st} \cdots \leq_{st} F_1$. Denote $\mathbb{K}_R = \{L: L \in \mathbb{K} \text{ and } l_1 \geq l_2 \geq \cdots l_n\}$.

Misra et al. [15] proved that, if $K, K' \in \mathbb{K}_R$, then

$$T_s(K) \geq_{st} T_s(K'), \text{ whenever } K' \geq_m K.$$

Li and Ding [15] showed, for $K, K' \in \mathbb{K}$ such that $l_i = l'_j$ and $l_j = l'_i$ for some $1 \leq i \leq j \leq n$, and $l_r = l'_r$ for $r \in \{1, \cdots, n\}/\{i, j\}$, it holds that

$$T_{k|n}(K) \geq_{st} T_{k|n}(K') \text{ if and only if } l_i < l_j.$$

This implies that one should allocate more redundancies to the component which is stochastically smaller. Then in order to obtain the optimal allocation policy, it is natural one should pay more attention to the set

$$\widetilde{\mathbb{K}_R} = \{L: L \in \mathbb{K} \text{ and } l_1 \leq l_2 \leq \cdots \leq l_n\}$$

for $K, K' \in \widetilde{\mathbb{K}_R}$, they proved, if $G \geq_{st} F_1$, then

$$T_{k|n}(K) \geq_{st} T_{k|n}(K') \text{ whenever } K' \geq_m K.$$

An example (Example 1) is also used to show that the condition $G \geq_{st} F_1$ cannot be dropped. Let $K^* = (l_1^*, \cdots, l_n^*) \in \mathbb{K}$ such that $|l_j^* - l_i^*| \leq 1$ for any pair $i \neq j$ and $\overline{K^*} = (\overline{l_1^*}, \cdots, \overline{l_n^*}) \in \widetilde{\mathbb{K}_R}$ such that $|l_j^* - l_i^*| \leq 1$ for any pair $i \neq j$. Obviously, K^* is not unique and $\overline{K^*}$ is unique. Then, for any $K \in \mathbb{K}$, if $G \geq_{st} F_1$, then

$$T_{k|n}(K) \leq_{st} T_{k|n}(\overline{K^*}),$$

i.e, $\overline{K^*}$ is the optimal allocation policy.

Problem 2.6 There should have stronger order results of series systems for the special case when $n = 2$, like likelihood ratio order.

Problem 2.7 What is the picture for the case when $G \leq_{st} F_n$ and can we obtain some results for this case?

2.3 Active redundancy at component level vs. system level

It is known that the performance of a coherent system consisting of independent components can be enhanced by putting spares to each of its components or by creating a duplicate system consisting of spares similar to the original coherent system. For example, there is a series system with n components and we have another n-component at hand as spares. In order to enhance the lifetime of the original series system, we can assemble these n spares either on the component level (Figure 1(a)) or on the system level (Figure 1(b)). Naturally, it is of great interest and importance to make sure which type is superior to the other.

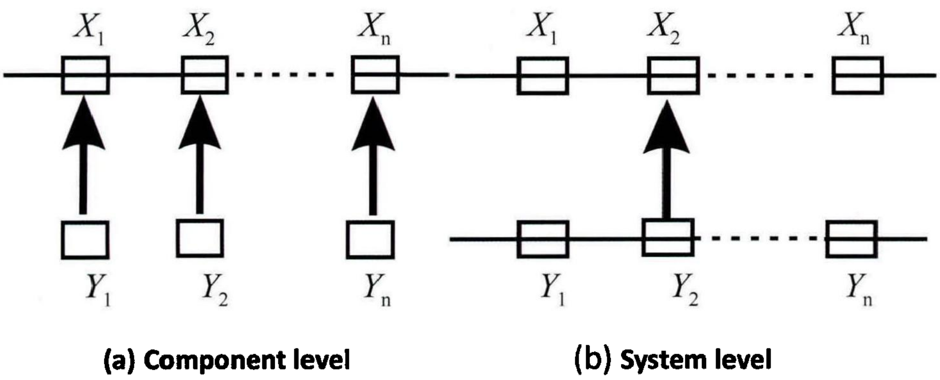

Figure 1 Component level versus system level.

Consider a series system with two components and two spares. Let X_1, X_2 be the lifetimes of components and Y_1, Y_2 be the lifetimes of spares. Denote

$$Q_1 = \min\{\max\{X_1, Y_1\}, \max\{X_2, Y_2\}\}$$

and

$$Q_2 = \max\{\min\{X_1, X_2\}, \min\{Y_1, Y_2\}\}$$

as the lifetimes of systems with redundancy at the component and system levels, respectively. In the case of matching spares, i. e., $X_i =_{st} Y_i, i=1,2$, the redundancy at the component level is better in the hazard rate ordering than the redundancy at the system level, that is, $Q_1 \geq_{hr} Q_2$ as shown in Boland and El-Neweihi [3]. An analogous result was proved for the reversed hazard rate ordering in Gupta and Nanda [12]. Brito et al. [8] further considered the case of matching spares and gave a condition under which the likelihood ratio ordering holds. Specifically, they proved, under this setup, if the hazard rates of X_1 and X_2 are proportional, then

$$Q_1 \geq_{lr} Q_2.$$

Gupta and Nanda [12] proved that $Q_1 \geq_{rh} Q_2$ in the case of non-matching spares, i. e., $X_1 =_{st} X_2$ and $Y_1 =_{st} Y_2$. Boland and El-Neweihi [3] showed with a counterexample that the hazard rate ordering does not always exist between them. In this regard, Brito et al. [8] proved that, if the reversed hazard rates of X_1 and Y_1 are proportional, then $Q_1 \geq_{hr} Q_2$.

In the component redundancy case, we allocate an active spare R_i to the component C_i, $i=1,\cdots,n$. Then the resultant coherent system, denoted by S_C, has lifetime $(\max(\boldsymbol{X},\boldsymbol{Y})) = \tau(\max(X_1,Y_1),\cdots,\max(X_n,Y_n))$. In the system redundancy case, we duplicate the coherent system $\emptyset$ with components $C_1,\cdots,C_n$ by $R_1,\cdots,R_n$ and make it available as an active redundant spare to the coherent system $\emptyset$. The resultant coherent system, denoted by S_S, has lifetime $max(\tau(X),\tau(Y))$. For general coherent systems, it is well known (cf. Barlow and Proschan [1]) that the component redundancy is better than the system redundancy in the sense of the usual stochastic ordering, i. e.,

$$\tau(\max(\boldsymbol{X},\boldsymbol{Y})) \geq_{st} \max(\tau(\boldsymbol{X}),\tau(\boldsymbol{Y})).$$

It would be very interesting to examine some other stronger stochastic orderings. In the literature, many researchers focused on this topic. In the case of matching spares, i. e., $X_i =_{st} Y_i$, $i=1,\cdots,n$. Boland and El-Neweihi [3] proved, for series systems, that the component redundancy is superior to the system redundancy in the sense of the hazard rate ordering, i. e.,

$$\tau(\max(\boldsymbol{X},\boldsymbol{Y})) \geq_{hr} \max(\tau(\boldsymbol{X}),\tau(\boldsymbol{Y})). \tag{7}$$

The above result can be readily extended to series-parallel systems (cf. Corollary 1-1 in Boland and El-Neweihi [3]), i. e., if $\tau(X) = \min_{1\leq j\leq n}(\max_{i\in A_j} X_i)$, where $A_1,\cdots A_m$ are disjoint, and $\bigcup_{j=1}^{m} A_j = \{1,\cdots n\}$, then (7) holds. When all components and spares are i. i. d., they also proved, for general coherent systems, if $ph'(p)/h(p)$ is non-increasing in p and $h(p) \leq p$ for all $p \in [0,1]$, where $h(p)$ denote the reliability function of the coherent system, then (1) also holds. It can be found that this result cannot be applied to the general k-out-of-n: G system, but they showed the result is true for the special case of 2-out-of-n: G system and left the general case for an open problem. Gupta and Nanda [12] obtained a similar result for the reversed hazard rate ordering. In fact, Singh and Singh [32] have solved the open problem posed by Boland and El-Neweihi [3] and established a stronger result in the sense of the likelihood ratio ordering for the general k-out-of-n:G system, that is, if all components and spares have i. i. d absolutely continuous lifetimes, then

$$\tau_{k|n:G}(\max(\boldsymbol{X},\boldsymbol{Y})) \geq_{lr} \max(\tau_{k|n:G}(\boldsymbol{X}),\tau_{k|n:G}(\boldsymbol{Y})).$$

Also, they remained an open problem that whether this result still hold for the matching case. For this open problem, the result for the two-component series system has been showed in Brito et al. [8] where they require that the hazard rates are proportional. Due to the complexity of distribution theory in this case, it is difficult to solve the above open problem for the general k-out-ot-n system. However, we have solved it for the case of series system under the exponential framework (Zhao et al. [39]).

Theorem 2.5 Let $X_1,\cdots,X_n$ be independent exponential lifetimes with respective parameters $\mu_1,\cdots,$

μ_n, and $Y_1, \cdots, Y_n$ be another independent set with respective parameters $\mu_1, \cdots, \mu_n$. Then,

$$\tau(\max(\boldsymbol{X},\boldsymbol{Y})) \geq_{lr} \max(\tau(\boldsymbol{X}), \tau(\boldsymbol{Y})).$$

Observe that the parallel system case is trivially true, and so we have the following further open problem for the general k-out-of-n system.

Problem 2.8 Let $X_1, \cdots, X_n$ be independent exponential lifetimes with respective parameters $\mu_1, \cdots, \mu_n$, and $Y_1, \cdots, Y_n$ be another independent exponential set with respective parameters $\mu_1, \cdots, \mu_n$. Then,

$$\tau_{k|n:G}(\max(\boldsymbol{X},\boldsymbol{Y})) \geq_{lr} \max(\tau_{k|n:G}(\boldsymbol{X}), \tau_{k|n:G}(\boldsymbol{Y})).$$

3 Standby redundancy

3.1 Allocation of one or two standby redundancies

Let us next introduce the standby redundancy model. In this case, one can decide which of these two allocations is better by carrying out stochastic comparisons on

$$T_1^s = \min\{X_1 + X, X_2, X_3, \cdots, X_n\}$$

and

$$T_2^s = \min\{X_1, X_2 + X, X_3, \cdots, X_n\}$$

for the series system, and

$$T_1^p = \max\{X_1 + X, X_2, X_3, \cdots, X_n\}$$

and

$$T_2^p = \max\{X_1, X_2 + X, X_3, \cdots, X_n\}$$

for the parallel system. Some results in the references pay attention to the allocation of one standby redundancy in a two-component series [parallel] system by stochastically comparing

$$S_1^s = \min\{X_1 + X, X_2\} \quad , \quad S_2^s = \min\{X_1, X_2 + X\}$$

and

$$S_1^p = \max\{X_1 + X, X_2\} \quad , \quad S_2^p = \max\{X_1, X_2 + X\}.$$

Similar to the case of active redundancy, here we also have two more general models. To decide which of these two allocations is better, one can make stochastic comparisons on

$$M_1^s = \min\{X_1 + Y_1, X_2, X_3, \cdots, X_n\}$$

and

$$M_2^s = \min\{X_1, X_2 + Y_2, X_3, \cdots, X_n\}$$

for the series system, and

$$M_1^p = \max\{X_1 + Y_1, X_2, X_3, \cdots, X_n\}$$

and

$$M_2^p = \max\{X_1, X_2 + Y_2, X_3, \cdots, X_n\}$$

for the parallel system. For the two-component series system, we define

$$N_1^s = \min\{X_1 + Y_1, X_2\} \quad \text{and} \quad N_2^s = \min\{X_1, X_2 + Y_2\},$$

and for the two-component parallel system, we define

$$N_1^p = \max\{X_1 + Y_1, X_2\} \quad \text{and} \quad N_2^p = \max\{X_1, X_2 + Y_2\}.$$

It is easy to see if Y_1 and Y_2 have identical distribution, then this model reduces to the case of allocation of one standby redundancy.

In the second model, one can compare these two allocation ways through stochastic comparisons on

$$H_1^s = \min\{X_1 + Y_1, X_2 + Y_2, X_3, \cdots, X_n\}$$

and

$$H_2^s = \min\{X_1 + Y_2, X_2 + Y_1, X_3, \cdots, X_n\}$$

for the series system, and

$$H_1^p = \max\{X_1 + Y_1, X_2 + Y_2, X_3, \cdots, X_n\}$$

and

$$H_2^p = \max\{X_1 + Y_2, X_2 + Y_1, X_3, \cdots, X_n\}$$

for the parallel system. For the two-component case, define

$$J_1^s = \min\{X_1+Y_1, X_2+Y_2\} \quad , \quad J_2^s = \min\{X_1+Y_2, X_2+Y_1\}$$

and

$$J_1^p = \max\{X_1+Y_1, X_2+Y_2\} \quad , \quad J_2^p = \max\{X_1+Y_2, X_2+Y_1\}.$$

Note if $P(Y_2=0)=1$, then this model reduces to the model of allocation of one standby redundancy.

For this topic, the related references include Boland et al. [5], Singh and Misra [30], Li and Hu [16], Li, Yan and Hu [17], Misra, Misra and Dhariyal [24], Zhao et al. [36] and the references therein.

Boland et al. [5] proved that

$$X_1 \le_{hr} X_2 \Leftrightarrow T_1^s \ge_{st} T_2^s \quad \text{and} \quad X_1 \le_{rh} X_2 \Leftrightarrow T_1^p \le_{st} T_2^p.$$

Singh and Misra [30] proved that

$$X_1 \le_{st} X_2 \Rightarrow T_1^s \ge_{sp} T_2^s \quad \text{and} \quad T_1^p \le_{sp} T_2^p.$$

Li and Hu [16] proved that if $X_1 \le_{icv} X_2$ and $X_1, X_3, \cdots, X_n$ have convex survival functions, then $T_1^s \ge_{sp} T_2^s$. They also proved that if $X_1 \le_{icv} X_2$ and if X_1 or X_2 has convex survival function, then $S_1^p \le_{sp} S_2^p$. Under the exponential framework, Zhao et al. [36] established the following result.

Theorem 3.1 Let X_1, X_2 and X be independent exponential random variables with parameters λ_1, λ_2 and λ, respectively. Denote $S_1 = \min\{\max\{X_1, X\}, X_2\}$ and $S_2 = \min\{X_1, \max\{X_2, X\}\}$. Then

$$\lambda_1 \ge \lambda_2 \Rightarrow S_1^s \ge_{lr} S_2^s.$$

Misra et al. [24] proved that

$$X_1 \le_{st} X_2 \quad \text{and} \quad X_1 \le_{icv} Y_1 \Rightarrow M_1^s \ge_{icv} M_2^s$$

and

$$X_1 \le_{st} X_2 \quad \text{and} \quad Y_1 \le_{icx} Y_2 \Rightarrow M_1^p \le_{icx} M_2^p.$$

They also proved that if X_1 or X_2 has an increasing failure rate, $\bar{F}_2(t)/\bar{F}_1(t)$ is log-convex in $t \in \Re_+$, then

$$X_1 \le_{hr} X_2 \quad \text{and} \quad Y_2 \le_{lr} Y_1 \Rightarrow N_1^s \ge_{hr} N_2^s.$$

Zhao et al. [38] recently established the following result.

Theorem 3.2 Let X_1, X_2, Y_1 and Y_2 be independent exponential random variables with parameters λ_1, λ_2, λ_1^* and λ_2^*, respectively. Suppose $\lambda_2 \le \lambda_1$ and $\lambda_1^* \le \lambda_2^*$. If $\lambda_1^2(\lambda_1^* + \lambda_2) - \lambda_2^2(\lambda_2^* + \lambda_1) \ge 0$, then $N_1^s \ge_{lr} N_2^s$.

Theorem 3.2 suggests that the standby redundancy with a longer expected lifetime should be allocated to the component with the shorter expected lifetime in a two-component series system. For the stochastic precedence order, Misra et al. [24] proved, if $X_1, X_3, \cdots, X_n$ (or $X_2, X_3, \cdots, X_n$) have convex survival functions on $\Re_+$, then

$$X_1 \le_{icv} X_2 \Rightarrow M_1^s \ge_{sp} M_2^s,$$

and if $X_1, X_3, \cdots, X_n$ (or $X_2, X_3, \cdots, X_n$) have concave survival functions on $\Re_+$, then

$$X_1 \le_{icx} X_2 \text{ and } Y_1 \le_{icx} Y_2 \Rightarrow M_1^p \le_{sp} M_2^p.$$

Li et al. [17] established that

$$X_1 \le_{sp} X_2, Y_1 \le_{sp} Y_2 \Leftrightarrow J_1^s \le_{sp} J_2^s$$

and

$$X_1 \ge_{sp} X_2, Y_1 \ge_{sp} Y_2 \Leftrightarrow J_1^p \ge_{sp} J_2^p.$$

For this case, Zhao et al. [37] established the following result for the likelihood ratio order.

Theorem 3.3 Let X_1, X_2, Y_1 *and* Y_2 be independent exponential random variables with parameters λ_1, λ_2, λ_1 and λ_2, respectively. Then, $J_1^s \le_{lr} J_2^s$.

For the parallel system, we have the following open problem.

Problem 3.1 Let X_1, X_2, Y_1 *and* Y_2 be independent exponential random variables with parameters λ_1, λ_2, λ_1 and λ_2, respectively. Then, $J_1^p \le_{lr} J_2^p$.

Li et al. [17] also proved that

$$X_1 \le_{st} X_2 \text{ and } Y_1 \le_{hr} Y_2 \Rightarrow H_1^s \le_{sp} H_2^s$$

and

$$X_1 \le_{st} X_2 \text{ and } Y_1 \le_{rh} Y_2 \Rightarrow H_1^p \ge_{sp} H_2^p.$$

Misra et al. [24] further proved that if $X_1, X_3, \cdots, X_n$ (or $X_2, X_3, \cdots, X_n$) have convex survival functions on $\Re_+$, then

$$X_1 \leq_{icv} X_2 \text{ and } Y_1 \leq_{hr} Y_2 \Rightarrow H_1^s \geq_{sp} H_2^s,$$

and if X_1 or X_2 has convex survival functions on $\Re_+$ and $X_i, i = 3, 4, \cdots n$, has a log-convex density on $\Re_+$, then

$$X_1 \leq_{icv} X_2 \text{ and } Y_1 \leq_{st} Y_2 \Rightarrow H_1^p \geq_{sp} H_2^p.$$

They also proved thatif $X_1, X_3, \cdots, X_n$ (or $X_2, X_3, \cdots, X_n$) have concave survival functions on $\Re_+$, then

$$X_1 \leq_{icx} X_2 \text{ and } Y_1 \leq_{rh} Y_2 \Rightarrow H_1^p \leq_{sp} H_2^p.$$

3.2 Standby redundancy at component level vs. system level

Consider a series system with two components and two standby spares. Denote

$$Q_1^s = \min\{X_1 + Y_1, X_2 + Y_2\}$$

and

$$Q_2^s = \min\{X_1, X_2\} + \min\{Y_1, Y_2\}$$

as the lifetimes of systems with redundancy at the component and system levels, respectively. In this part, we will use similar notations to those in Section 2.3. For the general coherent system, in component redundancy case, we allocate a standby spare R_i to the component C_i, $i = 1, \cdots, n$. Then the resultant system $\emptyset$, denoted by T_C, has lifetime $\tau(\boldsymbol{X}+\boldsymbol{Y}) = \tau(X_1+Y_1, \cdots, X_n+Y_n)$. In the system redundancy case, we duplicate the coherent system $\emptyset$ with components $C_1, \cdots, C_n$ by $R_1, \cdots, R_n$ and make it available as a standby redundant spare to the coherent system$\emptyset$. The resultant coherent system, denoted by T_S, has lifetime $\tau(\boldsymbol{X}) + \tau(\boldsymbol{Y})$. A well-known principle among design engineers states that redundancy at the component level is always better than at the system level. This statement, however, is not true in general for the standby redundancy case though it is for the active redundancy case. Boland and El-Neweihi [3] gave the conclusion that, in the sense of the usual stochastic order, redundancy at the component level is better than redundancy at the system level for series systems, while the reverse is true for the parallel systems. For the general k-out-of-n systems, there is no any comparison result. Boland and El-Neweihi [3] also showed that, for the series systems, if all random variables are exponentially i. i. d, then the usual stochastic order can be strengthened to the hazard rate order. Meng [20] further gave the equivalent relationship.

In this regard, we have established some new results when the original components and spares have independent but nonidentical exponential distributed lifetimes (Zhao et al. [39]). Specifically, for the standby redundancy of series systems, if either $X_i =_{st} Y_i$ ($i = 1, \cdots, n$) or $X_1 =_{st} \cdots =_{st} X_n$ and $Y_1 =_{st} \cdots =_{st} Y_n$, it is established that

$$\min(X_1 + Y_1, \cdots X_n + Y_n) \geq_{lr} \min(X_1, \cdots X_n) + \min(Y_1, \cdots Y_n).$$

For the standby redundancy of parallel systems, if $X_1 =_{st} X_2$ and $Y_1 =_{st} Y_2$, we established

$$\max(X_1 + Y_1, X_2 + Y_2) \leq_{lr} \max(X_1, X_2) + \max(Y_1, Y_2),$$

and if $X_i =_{st} Y_i$ ($i = 1, 2$), we proved

$$\max(X_1 + Y_1, X_2 + Y_2) \leq_{st} \max(X_1, X_2) + \max(Y_1, Y_2).$$

Problem 3.2 Are there similar results for the case when all the original components and standby redundancies have different distributions?

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